The Wigner function of a *q*-deformed harmonic oscillator model

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Abstract
The phase space representation for a *q*-deformed model of the quantum harmonic oscillator is constructed. We have found explicit expressions for both the Wigner and Husimi distribution functions for the stationary states of the *q*-oscillator model under consideration. The Wigner function is expressed as a basic hypergeometric series, related to the Al-Salam-Chihara polynomials. It is shown that, in the limit case $h \to 0$ ($q \to 1$), both the Wigner and Husimi distribution functions reduce correctly to their well-known non-relativistic analogues. Surprisingly, examination of both distribution functions in the *q*-deformed model shows that, when $q \ll 1$, their behaviour in the phase space is similar to the ground state of the ordinary quantum oscillator, but with a displacement towards negative values of the momentum. We have also computed the mean values of the position and momentum using the Wigner function. Unlike the ordinary case, the mean value of the momentum is not zero and it depends on *q* and *n*. The ground-state-like behaviour of the distribution functions for excited states in the *q*-deformed model opens quite new perspectives for further experimental measurements of quantum systems in the phase space.

PACS numbers: 03.65.−w, 03.65.Ta, 42.50.Dv, 02.30.Gp

1. Introduction

The harmonic oscillator concept occupies a central position in science and engineering due to its simplicity and exact solubility in both classical and quantum descriptions. Today it appears in mechanics, electromagnetism, electronics, optics, acoustics, astronomy, nuclear theory, to name just a few [1].

In the quantum approach, the harmonic oscillator is one of the exactly solvable problems studied in detail due to its considerable physical interest and applicability [2]. To examine the applicability of a system modelled by a harmonic oscillator, one wishes to compare theoretical results about position and momentum with measurements. In practical quantum applications however, one is faced with measurability problems of the quantum system states due to the uncertainty principle of quantum physics. For this reason, it is common to work with area elements in phase space, whose size is not smaller than Planck's constant, and obtain an expression for the joint probability distribution function of momentum and position. Then, having a certain method for the measurement of this quantity, the gathered results will provide a complete characterization of the quantum system under consideration and allow the use of more transparent classical language to examine its properties.

The Wigner distribution function is the main theoretical tool to construct the phase space of a quantum system. It is the closest quantum mechanical analogue of the classical probability distribution over the phase space, i.e. $\lim_{\hbar \to 0} W(p, x) = \rho(p, x)$. This distribution function was first proposed by Wigner to study the quantum correction for a thermodynamic equilibrium [3]. Quantum harmonic oscillators are one of first systems where analytical expressions for the Wigner distribution function were calculated, for stationary states as well as for states of the thermodynamic equilibrium. These expressions are important due to experimental measurements done in the past two decades [4, 5].

For example, the novel experimental method of optical homodyne tomography allows the measurement of the Wigner function for squeezed, vacuum, one- and two-photon states [4, 6, 7]. Note also experiments where measurements of the Wigner function of Fock states were performed on vibrational states of a trapped Be^+ ion [8]. In other words, advanced measurement techniques for the Wigner function changed its status from an 'only theoretically computable expression' to that of an 'also directly measurable quantity'.

All of this, however, is still not applicable to generalizations of the quantum harmonic oscillator, in particular the so-called q-deformed harmonic oscillator. The origin of q-deformed oscillators can be traced back to the work of Iwata [9], who generalized the Heisenberg commutation relation and solved the relevant eigenvalue problem. Later on, in the 1970s q-deformed generalizations of the harmonic oscillator were quite popular and since the end of the 1980s they are back into fashion. It is interesting to observe that q-deformed harmonic oscillators models are also simple and exactly solvable. Moreover, the appearance of an extra parameter q gives rise to additional application opportunities compared to the non-relativistic harmonic oscillator. Deformed oscillators have found applications in the theory of quons [10], statistical physics [11], generalized thermodynamics [12] and in models describing a small violation of the Pauli exclusion principle [13], to name a few. However, in spite of these applications, many aspects of the q-oscillator model have not been developed. One of the main reasons restricting its wide applications is the absence of an exact expression in terms of phase space. In this context, it is necessary to note the work in [14–16], where the possibility of obtaining the Wigner function associated with the q-Heisenberg commutation relation is discussed. However, in spite of attempts to derive an analytical expression of the Wigner function for the q-deformed harmonic oscillator models, up till now this goal was not achieved. The main problem is related to difficulties for the right definition of the momentum and position operators and to transformation formulae between them in the q-case.

If one imagines modern quantum physics as a puzzle, then due to the absence of an explicit description of q-deformed harmonic oscillators in phase space, some pieces of the puzzle are missing. In this paper, we examine some missing pieces of this puzzle and present explicit expressions for the Wigner and Husimi distribution functions for the stationary states of a q-deformed harmonic oscillator model. We also analyse these expressions for various

values of the parameter q and prove that they reduce to the well-known expressions for the non-relativistic harmonic oscillator in the limit $q \to 1$.

Our paper is structured as follows: in section 2, we provide basic information about the definition of the Wigner function and its Gauss smoothed analogue. We present their explicit expressions for the stationary states of the non-relativistic linear harmonic oscillator. Section 3 is devoted to a model of the q-deformed oscillator, whose wavefunctions are expressed in coordinate and momentum spaces by Rogers–Szegő and Stieltjes–Wigert polynomials, respectively. We present explicit expressions for the Wigner and Husimi distribution functions for the stationary states of the q-deformed oscillator in section 4. Discussions and conclusions are given in sections 5 and 6.

To end this introduction, we collect some common notation and known results for q-series. Many of the expressions encountered in this paper will be expressible in terms of basic hypergeometric series (q-series), for which we use the standard notation of Gasper and Rahman [17]. The q-shifted factorial is defined as

$$(a; q)_0 = 1 \qquad \text{and} \qquad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \qquad \text{for} \quad n = 1, 2, \ldots, \infty,$$

and $(a_1, \ldots, a_m; q)_n = (a_1; q)_n \cdots (a_m; q)_n$. A basic hypergeometric series is defined as

$$_r\phi_s \begin{pmatrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{pmatrix}; q, z = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, b_2, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} z^k.$$

In most cases, we will have that $r = s + 1$. Moreover, a basic hypergeometric series is terminating if one of the parameters a_j $(j = 1, \ldots, r)$ equals q^{-n} with n a nonnegative integer. The single most important summation formula for basic hypergeometric series is the q-binomial theorem:

$$_1\varphi_0 \begin{pmatrix} a \\ - \end{pmatrix}; q, z = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k = \frac{(az; q)_\infty}{(z; q)_\infty}, \qquad |z| < 1, \quad |q| < 1. \quad (1.1)$$

In the terminating case, i.e. when $a = q^{-n}$, this reduces to

$$_1\varphi_0 \begin{pmatrix} q^{-n} \\ - \end{pmatrix}; q, z = \frac{(q^{-n}z; q)_\infty}{(z; q)_\infty} = (q^{-n}z; q)_n,$$

and there are no longer any convergence conditions.

It will also prove convenient to have a notation for the q-binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

and for the q-numbers:

$$[a]_q = \frac{1 - q^a}{1 - q}, \qquad q \neq 1.$$

2. The Wigner distribution function

The Wigner function for stationary states of a quantum system can be obtained from its wavefunctions $\psi_n(x)$ or $\tilde{\psi}_n(p)$ (in the position or momentum representation) by the well-known definition [18]

$$W_n(p, x) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \psi_n^* \left(x - \frac{1}{2}x' \right) \psi_n \left(x + \frac{1}{2}x' \right) e^{-ipx'/\hbar} \, dx', \qquad (2.1a)$$

$$W_n(p, x) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \tilde{\psi}_n^* \left(p - \frac{1}{2} p' \right) \tilde{\psi}_n \left(p + \frac{1}{2} p' \right) e^{\mathrm{i}xp'/\hbar} \, \mathrm{d}p'. \qquad (2.1b)$$

It allows one to calculate the quantum average of a physical quantity f by the formula

$$\bar{f}_n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p, x) W_n(p, x) \, \mathrm{d}p \, \mathrm{d}x, \qquad (2.2)$$

where $f(p, x)$ is the Weyl symbol of the operator $\hat{f}(\hat{p}, \hat{x})$ [19].

The following explicit expression of the Wigner function for the stationary states of the non-relativistic linear harmonic oscillator is well known [19, 20]:

$$W_n^{\mathrm{HO}}(p, x) = \frac{(-1)^n}{\pi\hbar} \exp\left[-\frac{2}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right] \cdot L_n \left(\frac{4}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right), \qquad (2.3)$$

where L_n are the Laguerre polynomials ([21] section 1.11).

The Wigner distribution function determined in such a way takes both negative and positive values, therefore it is only a quasiprobability distribution function of p and x. It is bounded by the restriction $|W_n(p, x)| \leqslant (\pi\hbar)^{-1}$ [22]. For this reason, a class of Gaussian smoothed distribution functions with nonnegative behaviour was introduced [23–25]:

$$\overline{W}_n(p, x) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{p'^2}{2\Delta_p^2} - \frac{x'^2}{2\Delta_x^2} \right) \cdot W_n(p + p', x + x') \, \mathrm{d}p' \, \mathrm{d}x' \geqslant 0, \qquad (2.4)$$

where $\Delta_p \Delta_x$ is a finite region of the phase plane. The simplest form of the Gauss smoothed Wigner function is when $\Delta_p \Delta_x = \hbar/2$, where (2.4) will be a so-called Husimi distribution function [26]:

$$\overline{W}_n(p, x) = \frac{1}{(2\pi)^{3/2}\hbar\Delta_x} \left| \int_{-\infty}^{\infty} \psi_n(x') \cdot \exp\left[-\frac{\mathrm{i}px'}{\hbar} - \frac{(x - x')^2}{4\Delta_x^2} \right] \mathrm{d}x' \right|^2. \qquad (2.5)$$

Equation (2.5) has the property that $\overline{W}_n(p, x)$ is restricted by the condition $0 \leqslant \overline{W}_n(p, x) \leqslant (\pi\hbar)^{-1}$.

The calculation of the distribution function of the non-relativistic harmonic oscillator with this formula, when the parameter Δ_x^2 is taken equal to $\hbar/2m\omega$ (simplest case), leads to the expression [19]

$$\overline{W}_n^{\mathrm{HO}}(p, x) = (2\pi\hbar n!)^{-1} \left[\frac{1}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right]^n \cdot \exp\left[-\frac{1}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right]. \qquad (2.6)$$

3. The *q*-deformed harmonic oscillator model

The *q*-deformed harmonic oscillator model considered in this paper was already developed in a number of papers [27–35]. It has the following creation and annihilation operators:

$$b^{\pm} = \pm \frac{\mathrm{i}}{\sqrt{1-q}} \, e^{\mp\lambda x^2} \left(\exp\left(\mp 2\mathrm{i}\lambda h x \right) - q^{1/2} \exp\left(-\frac{\mathrm{i}h}{2} \partial_x \right) \right) e^{\pm\lambda x^2}, \qquad (3.1)$$

with $\lambda = \frac{m\omega}{2\hbar}$. Herein, h is a deformation parameter related to a finite-difference method with respect to x and

$$q = e^{-\lambda h^2}.$$

In the rest of this paper, q will always have the value $e^{-\lambda h^2}$.

Note that an operator of the form $\exp(a\partial_x)$, where a is an arbitrary complex number, acts as follows on functions of x:

$$\exp(a\partial_x)f(x) = f(x + a).$$

In the limit $h \to 0$ (or $q \to 1$), the operators (3.1) reduce to the well-known non-relativistic harmonic oscillator creation and annihilation operators [34]:

$$\lim_{h \to 0} b^\pm = a^\pm = \mp \frac{1}{2\sqrt{\lambda}} \, \mathrm{e}^{\pm\lambda x^2} \frac{\partial}{\partial x} \, \mathrm{e}^{\mp\lambda x^2}.$$

It is easily verified that

$$[b^-, b^+]_q = b^- b^+ - q b^+ b^- = 1.$$

The Hamiltonian operator H of the model is then

$$H = \hbar\omega\left(\sqrt{q}\, b^+ b^- + \left[\tfrac{1}{2}\right]_q\right).$$

The wavefunctions for this model in the x-representation [36] are related to the Rogers–Szegö polynomials $\mathcal{H}_n(x|q)$ ([37], chapter 17). More in particular, one has

$$\psi_n^{q\mathrm{HO}}(x) = c_n \mathcal{H}_n(-\mathrm{e}^{-2\mathrm{i}\lambda hx}|q)\,\mathrm{e}^{-\lambda x^2}, \tag{3.2}$$

where the normalization constant c_n is given by

$$c_n = \left(\frac{2\lambda}{\pi}\right)^{1/4} (-\mathrm{i})^n q^{n/2} (q;q)_n^{-1/2}.$$

One can see that in the limit for $h \to 0$ one recovers the wavefunctions for the non-relativistic quantum mechanical harmonic oscillator in the x-representation:

$$\psi_n^{q\mathrm{HO}}(x) \overset{h \to 0}{\to} \psi_n^{\mathrm{HO}}(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi/2\lambda}}} H_n(\sqrt{2\lambda}\,x) \cdot \mathrm{e}^{-\lambda x^2}. \tag{3.3}$$

The above limit can be verified by employing the following known limit relation between the Rogers–Szegö and Hermite polynomials [34, 36]:

$$\lim_{\alpha \overset{>}{\to} 0} \left(-\mathrm{i}\sqrt{\frac{2\tilde{q}}{1 - \tilde{q}}}\right)^n \mathcal{H}_n(-\mathrm{e}^{-2\mathrm{i}\alpha y}|\tilde{q}) = H_n(y), \qquad \tilde{q} = \mathrm{e}^{-2\alpha^2}, \qquad \alpha > 0. \tag{3.4}$$

The wavefunctions in the momentum and space representations are related through a Fourier transform:

$$\tilde{\psi}_n^{q\mathrm{HO}}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi_n^{q\mathrm{HO}}(x) \exp\left(-\frac{\mathrm{i}xp}{\hbar}\right) \mathrm{d}x.$$

Those in the p-representation are expressed through the Stieltjes–Wigert polynomials $S_n(x;q)$:

$$\tilde{\psi}_n^{q\mathrm{HO}}(p) = \frac{c_n (q;q)_n}{\sqrt{2\lambda\hbar}} S_n\left(q^{-1/2} \exp\left(-\frac{hp}{\hbar}\right); q\right) \exp\left(-\frac{p^2}{4\lambda\hbar^2}\right), \tag{3.5}$$

where we have used the notation and normalization of ([21], section 3.27) for the Stieltjes–Wigert polynomials $S_n(x;q)$.

Also here, one recovers the correct expression of the wavefunctions for the non-relativistic quantum mechanical harmonic oscillator in the p-representation when $h \to 0$:

$$\tilde{\psi}_n^{q\mathrm{HO}}(p) \overset{h \to 0}{\to} \tilde{\psi}_n^{\mathrm{HO}}(p) = \frac{(-\mathrm{i})^n}{\sqrt{2^n n! \sqrt{\pi \cdot m\omega\hbar}}} H_n(p/\sqrt{m\omega\hbar}) \cdot \exp\left(-\frac{p^2}{2m\omega\hbar}\right), \tag{3.6}$$

which follows from the limit relation between the Stieltjes–Wigert and Hermite polynomials [34, 36]:

$$\lim_{\substack{\alpha \to 0 \\ >}} \left(\frac{2\tilde{q}}{1-\tilde{q}}\right)^{n/2} (\tilde{q};\tilde{q})_n \, S_n(\tilde{q}^{-1/2} \, e^{-2\alpha y};\tilde{q}) = H_n(y), \qquad \tilde{q} = e^{-2\alpha^2}, \qquad \alpha > 0. \quad (3.7)$$

Thanks to the orthogonality relations for the Rogers–Szegö and Stieltjes–Wigert polynomials on the real axis [36],

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathcal{H}_n(-e^{-2i\alpha y}|\tilde{q}) \cdot \mathcal{H}_m(-e^{2i\alpha y}|\tilde{q}) \cdot e^{-y^2} \, dy = \frac{(\tilde{q};\tilde{q})_n}{\tilde{q}^n} \delta_{nm}, \quad (3.8)$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} S_n(\tilde{q}^{-1/2} \, e^{-2\alpha y};\tilde{q}) \cdot S_m(\tilde{q}^{-1/2} \, e^{-2\alpha y};\tilde{q}) \cdot e^{-y^2} \, dy = \frac{1}{(\tilde{q};\tilde{q})_n \tilde{q}^n} \delta_{nm}, \quad (3.9)$$

both wavefunctions (3.2) and (3.5) are also orthonormal:

$$\int_{-\infty}^{\infty} \psi_n^{*\,qHO}(x) \psi_m^{qHO}(x) \, dx = \delta_{nm}, \qquad \int_{-\infty}^{\infty} \tilde{\psi}_n^{*\,qHO}(p) \tilde{\psi}_m^{qHO}(p) \, dp = \delta_{nm}. \quad (3.10)$$

Finally, let us mention that the spectrum of the Hamiltonian H is given by

$$E_{n,q} = \hbar\omega \left[n + \frac{1}{2}\right]_q, \qquad \text{with} \quad n = 0, 1, 2, \ldots.$$

4. Computation of distribution functions

4.1. The Wigner distribution function for stationary states

The Wigner function for stationary states of the q-oscillator can be computed using (2.1a) and (3.2) or (2.1b) and (3.5). Substitution of (3.2) into (2.1a) leads to the following integral:

$$W_{n,q}(p,x) = \frac{|c_n|^2}{2\pi\hbar} \int_{-\infty}^{\infty} \mathcal{H}_n(-\exp(i\lambda h(2x-x'))|q)\mathcal{H}_n(-\exp(-i\lambda h(2x+x'))|q)$$
$$\times \exp\left(-\lambda\left(x-\frac{x'}{2}\right)^2 - \lambda\left(x+\frac{x'}{2}\right)^2\right) \exp\left(-ipx'/\hbar\right) dx'. \quad (4.1)$$

Here, the Rogers–Szegö polynomials have the following explicit expression:

$$\mathcal{H}_n(x|\tilde{q}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\tilde{q}} \left(\frac{x}{\tilde{q}^{1/2}}\right)^k, \qquad \text{with} \quad 0 < \tilde{q} < 1. \quad (4.2)$$

Using this expression, one writes the Wigner function (4.1) explicitly as

$$W_{n,q}(p,x) = \frac{1}{2\pi\hbar} |c_n|^2 \sum_{k,s=0}^{n} (-1)^{k+s} q^{-(k+s)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ s \end{bmatrix}_q$$
$$\times \int_{-\infty}^{\infty} \exp\left(-\lambda\left(x-\frac{x'}{2}\right)^2 - \lambda\left(x+\frac{x'}{2}\right)^2\right)$$
$$\times \exp\left(2i\lambda h\left(x-\frac{x'}{2}\right)k\right) \exp\left(-2i\lambda h\left(x+\frac{x'}{2}\right)s\right) \exp\left(-ipx'/\hbar\right) dx'.$$

Next, we use the Gaussian integral:

$$\int_{-\infty}^{\infty} \exp\left(-(a_2 x^2 + a_1 x + a_0)\right) dx = \sqrt{\frac{\pi}{a_2}} \exp\left(-a_0 + a_1^2/4a_2\right), \qquad \text{when}$$
$$a_2 > 0 \qquad \text{and} \qquad a_1, a_0 \in \mathbb{C}. \quad (4.3)$$

After some trivial simplifications, one then finds the following double sum expression for the Wigner distribution function:

$$W_{n,q}(p,x) = \frac{1}{\pi\hbar} \frac{q^n}{(q;q)_n} \exp\left(-2\lambda x^2 - p^2/2\lambda\hbar^2\right)$$

$$\times \sum_{k,s=0}^{n} (-1)^{k+s} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ s \end{bmatrix}_q q^{\binom{k}{2}+\binom{s}{2}+ks} e^{-ka^*-sa}, \tag{4.4}$$

with

$$a = \frac{hp}{\hbar} + 2i\lambda hx. \tag{4.5}$$

This expression can be further reduced to a single sum by applying the q-binomial theorem once. Indeed, we have

$$W_{n,q}(p,x) = \frac{1}{\pi\hbar} \frac{q^n}{(q;q)_n} \exp\left(-\frac{2}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right)\right) \sum_{s=0}^{n} (-1)^s \begin{bmatrix} n \\ s \end{bmatrix}_q q^{\binom{s}{2}} e^{-sa}$$

$$\times {}_1\varphi_0\left(\begin{matrix} q^{-n} \\ - \end{matrix}; q, q^{s+n} e^{-a^*}\right)$$

$$= \frac{1}{\pi\hbar} \frac{q^n}{(q;q)_n} \exp\left(-\frac{2}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right)\right) \sum_{s=0}^{n} (-1)^s \begin{bmatrix} n \\ s \end{bmatrix}_q q^{\binom{s}{2}} e^{-sa}(q^s e^{-a^*}; q)_n$$

$$= \frac{1}{\pi\hbar} \frac{q^n}{(q;q)_n} \exp\left(-\frac{2}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right)\right)(e^{-a^*}; q)_n$$

$$\times {}_2\varphi_1\left(\begin{matrix} q^{-n}, q^n e^{-a^*} \\ e^{-a^*} \end{matrix}; q, q^n e^{-a}\right).$$

Applying now the following transformation formula ([17], III.7)

$${}_2\varphi_1\left(\begin{matrix} \tilde{q}^{-n}, b \\ c \end{matrix}; \tilde{q}, z\right) = \frac{(c/b; \tilde{q})_n}{(c; \tilde{q})_n} {}_3\varphi_2\left(\begin{matrix} \tilde{q}^{-n}, b, bz\tilde{q}^{-n}/c \\ b\tilde{q}^{1-n}/c, 0 \end{matrix}; \tilde{q}, \tilde{q}\right),$$

leads to the following expression:

$$W_{n,q}(p,x) = \frac{(-1)^n}{\pi\hbar} q^{-\binom{n}{2}} \exp\left(-\frac{2}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right)\right) {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^n e^{-a}, q^n e^{-a^*} \\ q, 0 \end{matrix}; q, q\right). \tag{4.6}$$

Using the classical definition of the Al-Salam-Chihara polynomials ([21], section 3.8)

$$Q_n(y; \alpha, \beta | \tilde{q}) = \frac{(\alpha\beta; \tilde{q})_n}{\alpha^n} {}_3\varphi_2\left(\begin{matrix} \tilde{q}^{-n}, \alpha e^{i\theta}, \alpha e^{-i\theta} \\ \alpha\beta, 0 \end{matrix}; \tilde{q}, \tilde{q}\right), \qquad y = \cos\theta,$$

the Wigner function (4.6) can formally be written as

$$W_{n,q}(p,x) = \frac{(-1)^n}{\pi\hbar} \frac{q^{n(n+1)/2}}{(q;q)_n} \cdot \exp\left(-n\frac{h}{\hbar}p\right) \cdot \exp\left(-\frac{2}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right)\right)$$

$$\times Q_n\left(\cos 2\lambda hx; q^n \exp\left(-\frac{h}{\hbar}p\right), q^{1-n} \exp\left(\frac{h}{\hbar}p\right) \Big| q\right). \tag{4.7}$$

Quite generally, Wigner functions for stationary states satisfy the following relation [18]:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{n,q}(p,x) \cdot W_{m,q}(p,x) \, dx \, dp = \frac{\delta_{nm}}{2\pi\hbar}. \tag{4.8}$$

To prove this relation in the current case, one can use the double sum expression of the Wigner function (4.4). After simple calculations, using the Gaussian integral (4.3), the computation reduces to checking that

$$\frac{q^{n+m}}{(q;q)_n(q;q)_m} \sum_{k=0}^{n} (q^n)^k \frac{(q^{-n};q)_k}{(q;q)_k} (q^{-k};q)_m \cdot \sum_{k'=0}^{m} (q^m)^{k'} \frac{(q^{-m};q)_{k'}}{(q;q)_{k'}} (q^{-k'};q)_n = \delta_{nm}. \quad (4.9)$$

This is now readily verified since, on the left-hand side, all terms except the one with $k = k' = m = n$ are zero.

From the $_3\varphi_2$ expression (4.6) for the Wigner distribution function, it is easy to verify that in the limit for $h \overset{>}{\to} 0$ one recovers the Wigner distribution function for the non-relativistic quantum harmonic oscillator.

First, using series expansion in h one easily sees that

$$\lim_{h \overset{>}{\to} 0} \frac{(q^n e^{-a};q)_k (q^n e^{-a^*};q)_k}{(q;q)_k} = \prod_{j=0}^{k-1} \lim_{h \overset{>}{\to} 0} \frac{(1 - q^{n+j} e^{-a})(1 - q^{n+j} e^{-a^*})}{1 - q^{j+1}}$$

$$= \prod_{j=0}^{k-1} \lim_{h \overset{>}{\to} 0} \left(\frac{\frac{4}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right)}{j+1} + \mathcal{O}(h) \right) = \frac{\left(\frac{4}{\hbar\omega} \right)^k \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right)^k}{k!}.$$

Secondly, using a termwise limit for the $_3\varphi_2$-series, one finds that

$$\lim_{h \overset{>}{\to} 0} W_{n,q}(p,x) = \frac{(-1)^n}{\pi\hbar} \exp \left(-\frac{2}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right)$$

$$\times \sum_{k=0}^{n} \lim_{h \overset{>}{\to} 0} \frac{(q^{-n};q)_k}{(q;q)_k} \frac{(q^n \exp(-a);q)_k (q^n \exp(-a^*);q)_k}{(q;q)_k} q^k$$

$$= \frac{(-1)^n}{\pi\hbar} \exp \left(-\frac{2}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right) \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{\left(\frac{4}{\hbar\omega} \right)^k \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right)^k}{k!}$$

$$= \frac{(-1)^n}{\pi\hbar} \exp \left(-\frac{2}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right) L_n \left(\frac{4}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right)$$

$$= W_n^{\text{HO}}(p,x).$$

4.2. The Husimi distribution function

The calculation of the Husimi distribution function (2.5) follows more or less the same lines as that of the calculation of the Wigner distribution function.

First, one computes

$$\int_{-\infty}^{\infty} \psi_n(x') \exp \left(-\mathrm{i}px'/\hbar - (x - x')^2 / 4\Delta_x^2 \right) \mathrm{d}x' \quad (4.10)$$

using expressions (3.2), (4.2) and the Gaussian integral (4.3). Then, one can apply the q-binomial theorem. Secondly, one conjugates this expression and multiplies both expressions, taking into account the constant factor in (2.5). Finally, one arrives at

$$\overline{W}_{n,q}(p,x) = \frac{1}{2\pi\hbar} q^n \frac{(e^{-a/2};q)_n (e^{-a^*/2};q)_n}{(q;q)_n} \exp \left(-\frac{1}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right), \quad (4.11)$$

with a given by (4.5).

Using the same kind of limit calculation as before, one sees that the limit $h \overset{>}{\to} 0$ (or $q \overset{<}{\to} 1$) yields

$$\lim_{h \overset{>}{\to} 0} \overline{W}_{n,q}(p, x) = \frac{1}{2\pi\hbar n!} \left[\frac{1}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right]^n \exp\left(-\frac{1}{\hbar\omega} \left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \right) \right),$$

which is indeed the correct expression for the Husimi distribution function (2.6).

5. Discussions

The explicit expressions obtained for the Wigner and Husimi distribution functions for the q-deformed harmonic oscillator allow us to explore in detail the behaviour of these distribution functions in the phase space as q varies. First, an analysis of the ground state ($n = 0$, vacuum state) shows that the behaviour of the non-relativistic harmonic oscillator and its q-generalization in the phase space is the same. In other words, there is no impact of the parameter q in the absence of photons, and the Wigner function of the ground state for the q-deformed oscillator is the well-known Gaussian centred at the phase space point $(0, 0)$.

For the excited quantum states the situation is rather different. In figures 1 and 2, we have given a density plot for the single ($n = 1$) and double ($n = 2$) photon states of the q-deformed harmonic oscillator in the phase space. For simplicity, we use the scale $m = \omega = \hbar = 1$. The first plot in each of the figures is the Wigner function for the non-relativistic quantum harmonic oscillator ($h = 0$ or $q = 1$). As long as $h \leqslant 1$ we find a phase space that is not so different from that of the non-relativistic harmonic oscillator, except for the occurrence of a new peak at the point $x = 0$ and $p < 0$. This value $h = 1$ is important, because most of the investigations of finite-difference generalizations of the Schrödinger equation propose to take a finite-difference step h to be equal to $\lambdabar = \hbar/mc$, i.e. the Compton wavelength of a particle with mass m (see, for example, [28, 38, 39]). In this case ($h = 1$) one has $\hbar\omega = mc^2$. After this point, as h increases ($h > 1$) or q decreases, one can see a significant impact of the parameter on the behaviour of the q-deformed oscillator in the phase space. First, one can see a displacement of the probability function peak towards negative values of the momentum. This can be understood by computing the mean values of the q-oscillator position and momentum.

The mean value of the oscillator position in stationary states requires the computation of the following integral:

$$\bar{x} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x W_{n,q}(p, x) \, dx \, dp. \tag{5.1}$$

Using the double sum expression of the Wigner function (4.4), one can see that the integrand is an odd function with respect to x, and therefore (5.1) is zero.

The mean value of the momentum for stationary states is given by a similar expression:

$$\bar{p} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p W_{n,q}(p, x) \, dx \, dp. \tag{5.2}$$

Again, one can use the double sum expression of the Wigner function (4.4) to compute this integral, leading to

$$\bar{p} = -nm\omega h. \tag{5.3}$$

So, the average of the momentum depends on the number of photons and on the finite-difference step h. This is the reason for the displacement of the probability function towards negative values of the momentum.
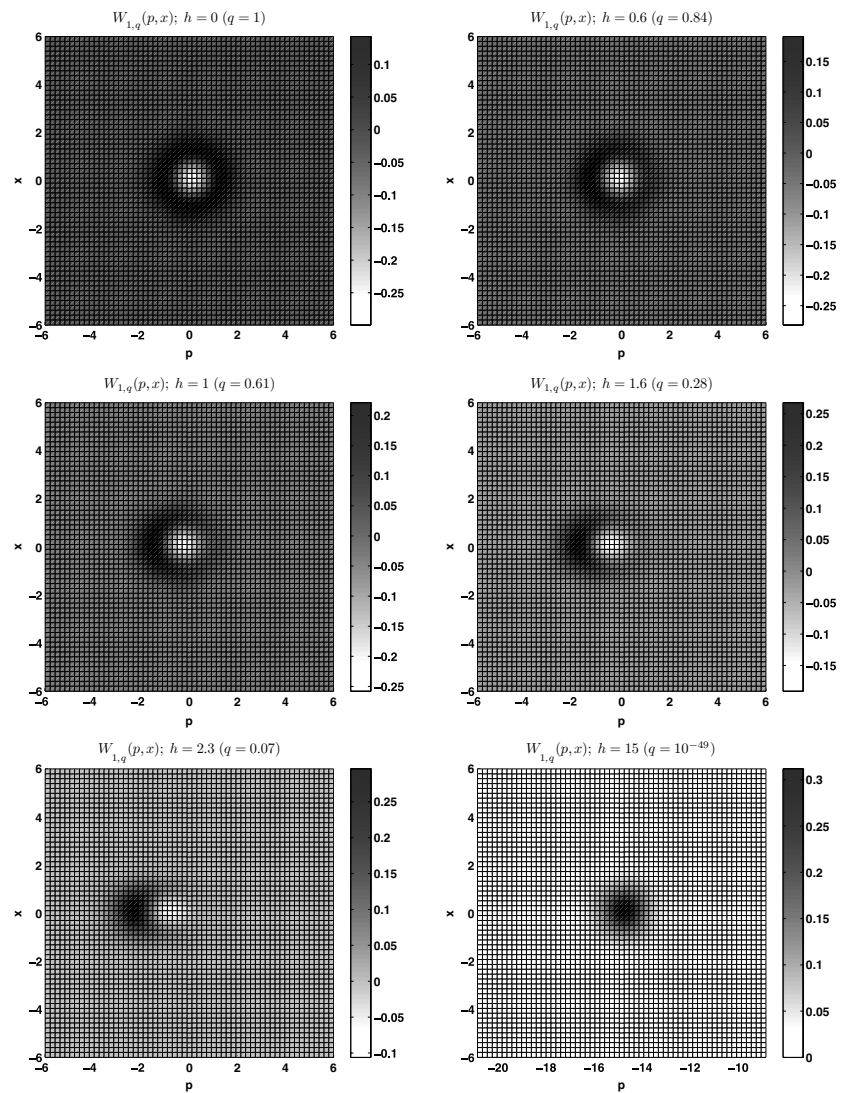
Figure 1. A density plot of the Wigner function of the single photon state for the q-deformed harmonic oscillator, for values of $h = 0, 0.6, 1, 1.6, 2.3, 15$ ($q = 1, 0.84, 0.61, 0.28, 0.07, 10^{-49}$) and $m = \omega = \hbar = 1$. The function changes dramatically when the value of the parameter q decreases. Observe also a displacement of the Wigner function due to the non-zero average of the q-oscillator momentum.

The last plots in figures 1 and 2 show a similar picture. In other words, the limit $q \to 0$ dramatically changes the behaviour of the Wigner function. The probability distribution function has apparently the form of a Gaussian, and its value does not seem to depend on the photon number n, only its displacement in phase space depends on n. This can be understood analytically by computing the limit $h \to +\infty$ (corresponding to $q \to 0$). In fact, for $n > 0$ and for fixed finite values of x and p one finds

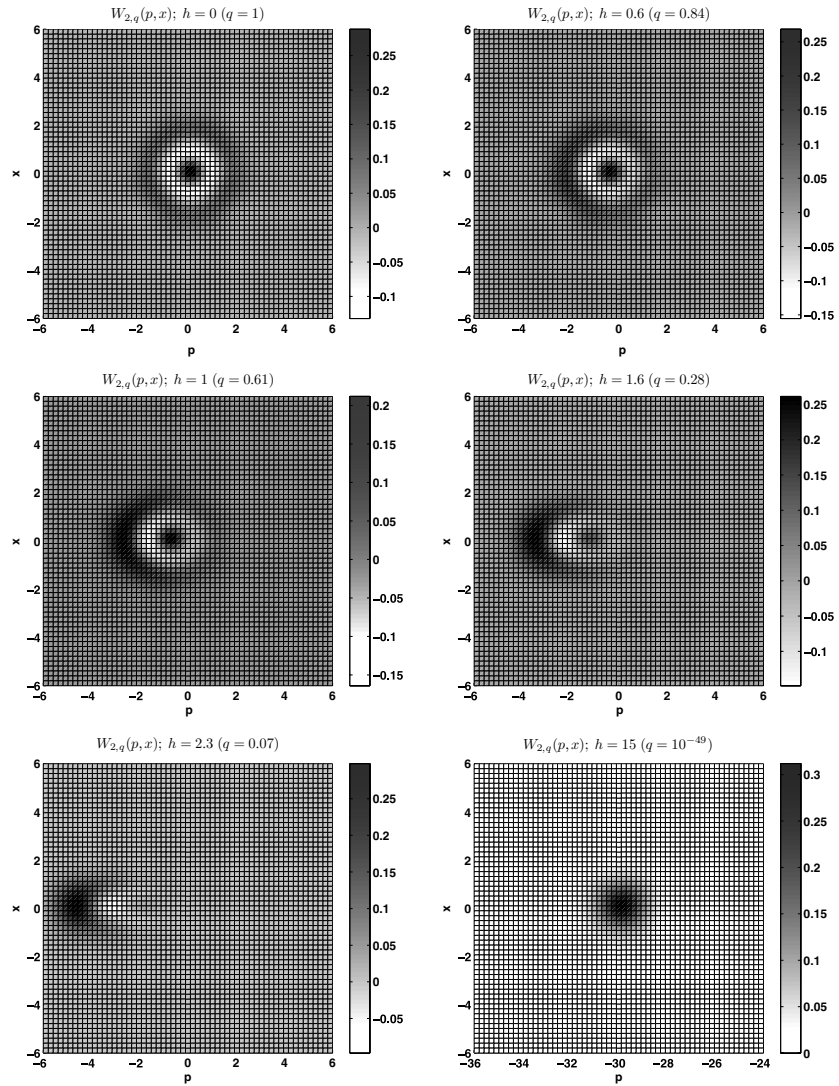$$\lim_{h \to +\infty} W_{n,q}(p, x) = 0; \tag{5.4}$$

Figure 2. A density plot of the Wigner function of the double photon state for the q-deformed harmonic oscillator, for values of $h = 0, 0.6, 1, 1.6, 2.3, 15$ ($q = 1, 0.84, 0.61, 0.28, 0.07, 10^{-49}$) and $m = \omega = \hbar = 1$. The function again changes dramatically when the value of the parameter q decreases. The displacement of the Wigner function due to the non-zero average of the q-oscillator momentum is clear, and one can also see the impact of the momentum average dependence on n.

on the other hand, for every q (or h)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{n,q}(p, x) \, \mathrm{d}x \, \mathrm{d}p = 1. \tag{5.5}$$

These relations indicate that $\lim_{h \to +\infty} W_{n,q}(p, x)$, for $n > 0$, actually behaves as Gaussian distribution function with the peak displaced towards $(-\infty, 0)$ in the (p, x)-plane.

In figures 3 and 4, we also present plots of the Husimi distribution function for the q-deformed harmonic oscillator, for $n = 1$ and $n = 2$. The behaviour of the plots in this
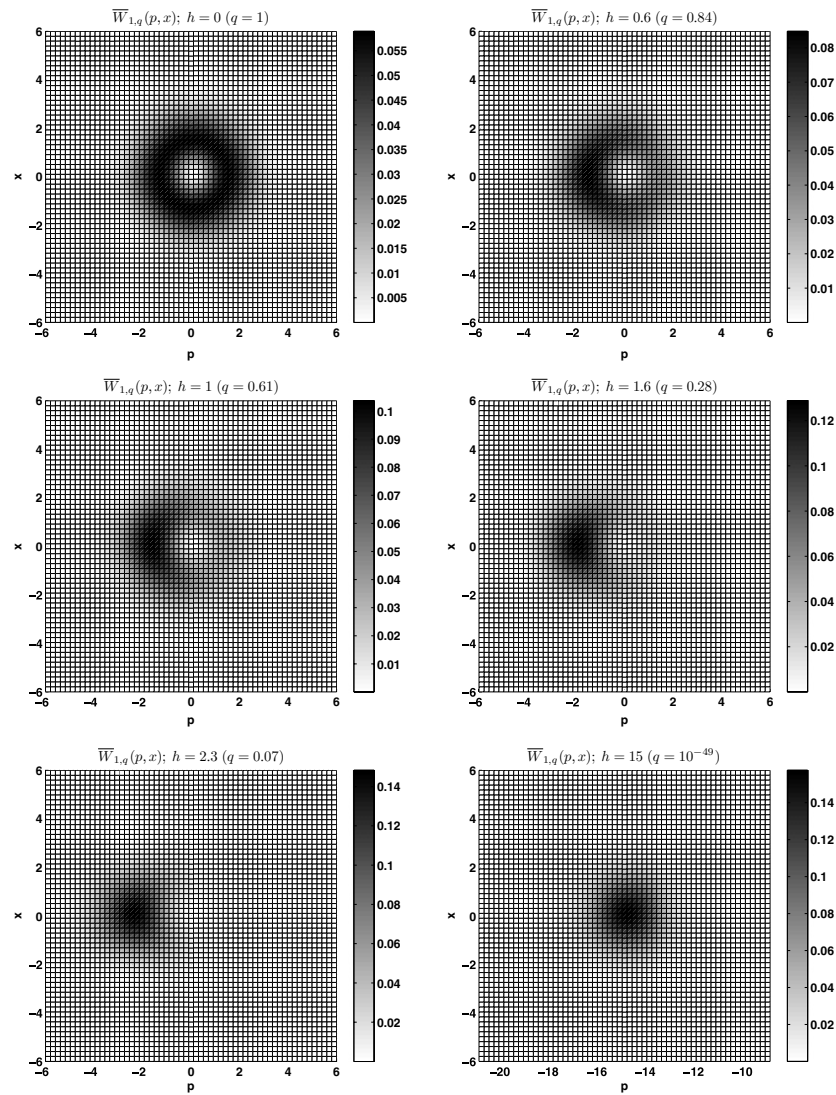
Figure 3. A density plot of the Husimi function of the single photon state for the q-deformed harmonic oscillator, for values of $h = 0, 0.6, 1, 1.6, 2.3, 15$ ($q = 1, 0.84, 0.61, 0.28, 0.07, 10^{-49}$) and $m = \omega = \hbar = 1$. The behaviour is similar to that of the Wigner function.

case is the same as for the Wigner distribution functions. Comparing the behaviour of the Wigner and Husimi distributions for small q-values ($0 < q \ll 1$) indicates that both of them are good approximations of a displaced vacuum state, but the Wigner function is defined more sharply than the Husimi function, allowing a more correct determination of the phase space probabilities of the quantum system under consideration. In fact, for the Husimi function one can verify that for large h-values, (4.11) is approximated by

$$\overline{W}_{n,q}(p, x) \approx \frac{1}{2\pi\hbar} \exp\left(-\frac{(p + nm\hbar\omega)^2}{2m\hbar\omega} - \frac{m\omega x^2}{2\hbar}\right).$$
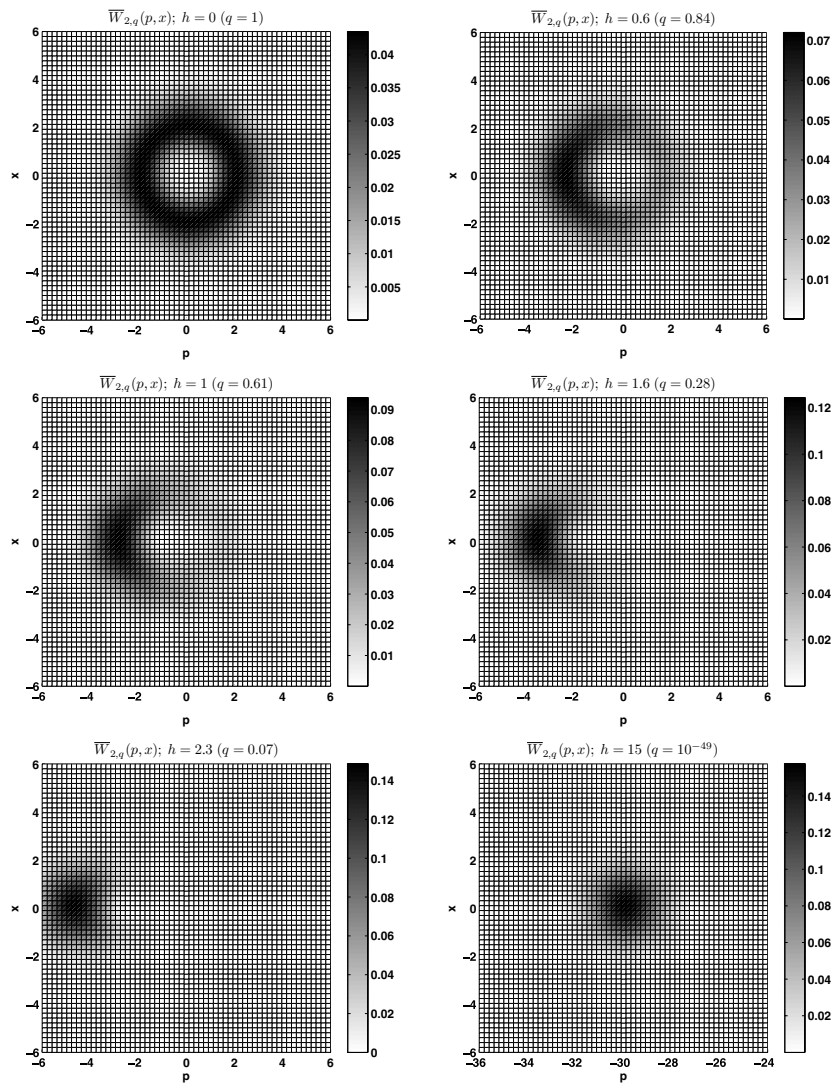
Figure 4. A density plot of the Husimi function of the double photon state for the *q*-deformed harmonic oscillator, for values of $h = 0, 0.6, 1, 1.6, 2.3, 15$ ($q = 1, 0.84, 0.61, 0.28, 0.07, 10^{-49}$) and $m = \omega = \hbar = 1$.

In a similar way, one can compute an approximation of the Wigner distribution function for large *h*-values from (4.6):

$$W_{n,q}(p, x) \approx \frac{1}{\pi\hbar} \exp\left(-\frac{(p + nm\hbar\omega)^2}{m\hbar\omega} - \frac{m\omega x^2}{\hbar}\right).$$

These expressions confirm the observed behaviour of the distribution functions as a displaced Gaussian when *h* is large (or when *q* is close to 0).

6. Conclusions

In recent years, one has seen an increased interest in the study of quantum systems in phase space, both analytically and experimentally. For example, nowadays the development of measurement technologies allows the construction of experimental setups to recover the Wigner distribution function of certain Fock states. The use of analytical expressions for the Wigner function allows for a more detailed study of quantum entanglement.

We have constructed the Wigner and Husimi distribution functions for the q-deformed linear harmonic oscillator, whose wavefunctions of the stationary states are expressed by means of Rogers–Szegö and Stieltjes–Wigert polynomials in the position and momentum representations. The analytical expression of the Wigner distribution function for the stationary states can formally be expressed by means of an Al-Salam-Chihara polynomial. The Husimi distribution function for the stationary states is simpler and is expressed by means of q-shifted factorials. Using proper limits for basic hypergeometric series we have shown that the Wigner and Husimi functions of the stationary states reduce to those for the non-relativistic quantum harmonic oscillator, when $q \rightarrow 1$.

Examination of both distribution functions in the q-model shows that, when q tends to 0, their behaviour in phase space is similar to the ground state of an ordinary quantum oscillator, but with a displacement of momentum towards negative values. We have computed the mean values of the position and momentum for the Wigner function and, unlike the ordinary case, the mean values of the momentum are not zero but depend on q and n. The ground-state-like behaviour of the distribution functions for excited states in the q-case opens up new perspectives for further experimental measurements of quantum systems in the phase space.

Acknowledgment

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